ITERATING LOWERING OPERATORS

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Abstract. For an algebraically closed base field of positive characteristic, an algorithm to construct some non-zero $\mathrm{GL}(n-1)$-high weight vectors of irreducible rational $\mathrm{GL}(n)$-modules is suggested. It is based on the criterion proved in this paper for the existence of a set A such that $S_{i,j}(A)f_{\mu,\lambda}$ is a non-zero $\mathrm{GL}(n-1)$-high weight vector, where $S_{i,j}(A)$ is Kleshchev's lowering operator and $f_{\mu,\lambda}$ is a non-zero $\mathrm{GL}(n-1)$-high weight vector of weight μ of the costandard $\mathrm{GL}(n)$-module $\nabla_n(\lambda)$ with highest weight λ.

1. Introduction

Classical lowering operators were introduced by Carter in [2]. Kleshchev used them in [5] to define generalized lowering operators. Following [1] and [4], we denote these operators by $S_{i,j}(A)$. Kleshchev's lowering operators are useful in constructing $\mathrm{GL}(n-1)$-high weight vectors from the first level of irreducible rational $\mathrm{GL}(n)$-modules. In fact, [5, Theorem 4.2] shows that every such vector has the form $S_{i,n}(A)v_+$, where v_+ is the $\mathrm{GL}(n)$-high weight vector. A natural idea is to continue to apply lowering operators $S_{i,j}(A)$ to the $\mathrm{GL}(n-1)$-high weight vectors already obtained in order to construct new $\mathrm{GL}(n-1)$-high weight vectors belonging to higher levels. For example, this method (for $j=n$) was used in [4] to construct all $\mathrm{GL}(n-1)$-high weight vectors of irreducible modules $L_n(\lambda)$, where λ is a generalized Jantzen-Seitz weight. The main aim of this paper is to find all $\mathrm{GL}(n-1)$-high weight vectors that can be constructed in this way (see Theorem 13 and Remark 2 for removing one node and Theorems 16 and 17 for moving one node).

Let K be an algebraically closed field of characteristic $p>0$ and $\mathrm{GL}(m)$ denote the group of invertible $m\times m$-matrices over K. We generally follow the notations of [4] and [1] and actually work with hyperalgebras rather than algebraic groups. For the connection between representations of the latter two, we refer the reader to [3]. Let $U(m,\mathbb{Z})$ denote the $\mathbb{Z}$-subalgebra of the universal enveloping algebra $U(m,\mathbb{C})$ of the Lie algebra $\mathfrak{gl}(m,\mathbb{C})$ that is generated by the identity element and

$$X_{i,j}^{(r)} := \frac{(X_{i,j})^r}{r!} \text{ for } 1\leqslant i,\ j\leqslant m, i\neq j \text{ and } r\geqslant 1;$$
$$\binom{X_{i,i}}{r} := \frac{X_{i,i}(X_{i,i}-1)\cdots(X_{i,i}-r+1)}{r!} \text{ for } 1\leqslant i\leqslant m \text{ and } r\geqslant 1,$$

where $X_{i,j}$ denotes the $m\times m$-matrix with 1 in the ij-entry and zeros elsewhere. We define the *hyperalgebra* $U(m)$ to be $U(m,\mathbb{Z})\otimes_{\mathbb{Z}} K$. For $1\leqslant i<j\leqslant m$ we denote by $E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ the images of $X_{i,j}^{(r)}$ and $X_{j,i}^{(r)}$

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respectively and for $1 \leqslant i \leqslant m$ denote by $\binom{H_i}{r}$ the image of $\binom{X_{i,i}}{r}$ under the above base change. If $r = 1$ then we omit the superscripts in the above definitions and write H_i for $\binom{H_i}{1}$. We also put $E_i^{(r)} := E_{i,i+1}^{(r)}$ and $F_{i,i}^{(r)} := 1$.

Let $U^0(m)$ denote the subalgebra of $U(m)$ generated by 1 and $\binom{H_i}{r}$ for $1 \leqslant i \leqslant m$ and $r \geqslant 1$ and $X^+(m)$ denote the set of integer sequences $(\lambda_1, \dots, \lambda_m)$ such that $\lambda_1 \geqslant \cdots \geqslant \lambda_m$. We say that a vector v of a $U(m)$-module *has weight* $\lambda \in X^+(m)$ if $\binom{H_i}{r} v = \binom{\lambda_i}{r} v$ for any $1 \leqslant i \leqslant m$ and $r \geqslant 1$. If moreover $E_i^{(r)} v = 0$ for any $1 \leqslant i < m$ and $r \geqslant 1$, then we say that v is a $U(m)$-*high weight vector*.

Throughout $[i..j]$, $(i..j]$, $[i..j)$, $(i..j)$ denote the sets $\{a \in \mathbb{Z} : i \leqslant a \leqslant j\}$, $\{a \in \mathbb{Z} : i < a \leqslant j\}$, $\{a \in \mathbb{Z} : i \leqslant a < j\}$, $\{a \in \mathbb{Z} : i < a < j\}$ respectively. For any condition $\mathcal{P}$, let $\delta_{\mathcal{P}}$ be 1 if $\mathcal{P}$ is true and 0 if it is false. Given a pair of integers (i, j), let $\operatorname{res}_p(i, j)$ denote $(i - j) + p\mathbb{Z}$, which is an element of $\mathbb{Z}/p\mathbb{Z}$. For any set $A \subset \mathbb{Z}$ and two integers $i \leqslant j$, let $A_{i..j} = \{a \in A : i < a < j\}$. If moreover $A \subset (i..j)$ then we put $F_{i,j}^A = F_{a_0,a_1} \cdots F_{a_k,a_{k+1}}$, where $A \cup \{i, j\} = \{a_0 < \cdots < a_{k+1}\}$. Thus $F_{i,j}^{\varnothing} = F_{i,j}$. For $i < j$ and $A \subset (i..j)$, the lowering operator $S_{i,j}(A)$ is defined as (see [1, Remark 4.8])

$$S_{i,j}(A) := \sum_{B \subset (i..j)} F_{i,j}^B H_{i,j}(A, B).$$

In this formula, $H_{i,j}(A, B)$ is the element of $U^0(m)$ obtained by evaluating the rational expression

$$\mathcal{H}_{i,j}(A, B) := \sum_{D \subset B \setminus A} (-1)^{|D|} \frac{\prod\limits_{t \in A} (x_t - x_{D_i(t)})}{\prod\limits_{t \in B} (x_t - x_{D_i(t)})},$$

where $D_i(t) = \max\{s \in D \cup \{i\} : s < t\}$, at $x_k := k - H_k$. Elements $H_{i,j}(A, B)$ are well defined, since $\mathcal{H}_{i,j}(A, B) \in \mathbb{Z}[x_i, \dots, x_{j-1}]$, which is proved in [1, Lemma 4.6(i)]. We additionally assume that $S_{i,i}(\varnothing) = 1$.

Quite easy proofs of all the properties of the operators $S_{i,j}(A)$ we need here can be found in [1], where the specialization $v \mapsto 1$ should be made.

In this paper, we work with costandard modules $\nabla_n(\lambda)$, where $\lambda \in X^+(n)$, and its non-zero $U(n-1)$-high weight vectors $f_{\mu,\lambda}$, where $\mu \in X^+(n-1)$ and $\lambda_i \geqslant \mu_i \geqslant \lambda_{i+1}$ for $1 \leqslant i < n$. If the last conditions hold we write $\mu \longleftarrow \lambda$. We also denote the element $f_{\bar{\lambda},\lambda}$, where $\bar{\lambda} = (\lambda_1, \dots, \lambda_{n-1})$, by f_λ. It is a $U(n)$-high weight vector generating the simple submodule $L_n(\lambda)$ of $\nabla_n(\lambda)$. The definitions of all these objects can be found in [4]. Moreover using [4, Lemma 2.6(ii)] and multiplication by a suitable power of the determinant representation of $\mathrm{GL}(n)$, we may assume that f_λ and $f_{\mu,\lambda}$, where $\mu \longleftarrow \lambda$ and $a_i := \sum_{s=1}^{i} (\lambda_s - \mu_s)$, are chosen so that $E_1^{(a_1)} \cdots E_{n-1}^{(a_{n-1})} f_{\mu,\lambda} = f_\lambda$.

2. Graph of sequences

For the remainder of this paper, we fix an integer $n > 1$ and weights $\lambda \in X^+(n)$, $\mu \in X^+(n-1)$ such that $\mu \longleftarrow \lambda$. For $i = 1, \dots, n-1$, we

put $a_i := \sum_{j=1}^{i}(\lambda_j - \mu_j)$. The following formulas can easily be checked by calculations in $U(n,\mathbb{Z})$.

Lemma 1. *Let $1 \leqslant i < j \leqslant n$, $1 \leqslant l < n$, $m \geqslant 1$ and $A \subset (i..j)$. We have*

(i) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)}$ *if $l \notin A \cup \{i\}$ and $l+1 \notin A \cup \{j\}$;*

(ii) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)} - F_{i,l}^{A_{i..l}} F_{l+1,j}^{A_{l+1..j}} E_l^{(m-1)}$ *if $l \in A \cup \{i\}$ and $l+1 \notin A \cup \{j\}$;*

(iii) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)} + F_{i,l}^{A_{i..l}} F_{l+1,j}^{A_{l+1..j}} E_l^{(m-1)}$ *if $l \notin A \cup \{i\}$ and $l+1 \in A \cup \{j\}$;*

(iv) $E_l^{(m)} F_{i,j}^A = F_{i,j}^A E_l^{(m)} + F_{i,l}^{A_{i..l}} (H_l - H_{l+1} + 1 - m) F_{l+1,j}^{A_{l+1..j}} E_l^{(m-1)}$ *if $l \in A \cup \{i\}$ and $l+1 \in A \cup \{j\}$.*

We shall use the abbreviation $E(i,j) = E_i \cdots E_j$. Let $1 \leqslant i \leqslant k \leqslant j \leqslant n$ and $A \subset (i..j)$. It follows from Lemma 1 that $E(k,j-1)S_{i,j}(A) = u_k E_k + \cdots + u_{j-1}E_{j-1} + M_{i,j}^k(A)$, where $u_k, \ldots, u_{j-1} \in U(n)$ and $M_{i,j}^k(A)$ is a linear combination of elements of the form $F_{i,k}^B H$, where $H \in U^0(n)$. In what follows, we stipulate that any not necessarily commutative product of the form $\prod_{i \in A} x_i$, where $A = \{a_1 < \cdots < a_m\} \subset \mathbb{Z}$, equals $x_{a_1} \cdots x_{a_m}$.

Lemma 2. *Given integers $1 \leqslant i_1 < j_1 < \cdots < i_{s-1} < j_{s-1} < i_s < j_s \leqslant n$, sets $A_1 \subset (i_1..j_1)$, $\ldots$, $A_s \subset (i_s..j_s)$ and integers $k_1, \ldots, k_s$ such that $i_t \leqslant k_t \leqslant j_t$ for $t = 1, \ldots, s$ and $j_s = n$ implies $k_s = n$, we put*

$$v = E(k_1, j_1 - 1)S_{i_1,j_1}(A_1) \cdots E(k_s, j_s - 1)S_{i_s,j_s}(A_s) f_{\mu,\lambda}.$$

Then we have

(i) $v = X_1 \cdots X_s f_{\mu,\lambda}$, *where each X_t is either $E(k_t, j_t - 1)S_{i_t,j_t}(A_t)$ or $M_{i_t,j_t}^{k_t}(A_t)$;*

(ii) $E_l^{(m)} v = 0$ *if $1 \leqslant l < n-1$ and $m \geqslant 2$;*

(iii) $E_l^{(m)} v = 0$ *if $m \geqslant 1$ and $l \in [1..n-1) \setminus \big([i_1..k_1) \cup \cdots \cup [i_s..k_s)\big)$;*

(iv) *If $i_t < k_t < n$ then*

$$E_{k_t - 1} v = \left(\prod_{r=1}^{t-1} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) E(k_t - 1, j_t - 1)S_{i_t,j_t}(A_t) \times \left(\prod_{r=t+1}^{s} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) f_{\mu,\lambda};$$

(v) *If $l \in [i_t..k_t - 1)$ then*

$$E_l v = c\left(\prod_{r=1}^{t-1} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) S_{i_t,l}((A_t)_{i_t..l}) \times E(k_t, j_t - 1)S_{l+1,j_t}((A_t)_{l+1..j_t}) \left(\prod_{r=t+1}^{s} E(k_r, j_r - 1)S_{i_r,j_r}(A_r)\right) f_{\mu,\lambda},$$

where $c = 0$ except the case $l \in A_t \cup \{i_t\}$, $l+1 \notin A_t$, in which $c = -1$.

Proof. (i) Applying Lemma 1, we prove by induction on t (starting from $t=s$) that

$$v = E(k_1,j_1-1)S_{i_1,j_1}(A_1)\cdots E(k_{t-1},j_{t-1}-1)S_{i_{t-1},j_{t-1}}(A_{t-1})$$
$$\times M^{k_t}_{i_t,j_t}(A_t)\cdots M^{k_s}_{i_s,j_s}(A_s)f_{\mu,\lambda}.$$

Using this formula for $t=1$, we obtain the required result by induction on s.

(ii), (iii) follow from part (i) for $X_t = M^{k_t}_{i_t,j_t}(A_t)$ and Lemma 1.

(iv) Applying part (i) (possibly for different parameters), we get

$$E_{k_t-1}v = E_{k_t-1}M^{k_1}_{i_1,j_1}(A_1)\cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})$$
$$\times E(k_t,j_t-1)S_{i_t,j_t}(A_t)\cdots E(k_s,j_s-1)S_{i_s,j_s}(A_s)f_{\mu,\lambda}$$
$$= M^{k_1}_{i_1,j_1}(A_1)\cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})E(k_t-1,j_t-1)S_{i_t,j_t}(A_t)$$
$$\times E(k_{t+1},j_{t+1}-1)S_{i_{t+1},j_{t+1}}(A_{t+1})\cdots E(k_s,j_s-1)S_{i_s,j_s}(A_s)f_{\mu,\lambda}.$$

Now the required formula follows from part (i).

(v) Since E_l and $E(k_t,j_t-1)$ commute in this case, we get by [1, 4.11(i),(ii)] and parts (i),(ii) of the current lemma that

$$E_lv = M^{k_1}_{i_1,j_1}(A_1)\cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})E(k_t,j_t-1)E_lS_{i_t,j_t}(A_t)$$
$$\times E(k_{t+1},j_{t+1}-1)S_{i_{t+1},j_{t+1}}(A_{t+1})\cdots E(k_s,j_s-1)S_{i_s,j_s}(A_s)f_{\mu,\lambda} =$$
$$cM^{k_1}_{i_1,j_1}(A_1)\cdots M^{k_{t-1}}_{i_{t-1},j_{t-1}}(A_{t-1})S_{i_t,l}((A_t)_{i_t..l})E(k_t,j_t-1)S_{l+1,j_t}((A_t)_{l+1..j_t})$$
$$\times E(k_{t+1},j_{t+1}-1)S_{i_{t+1},j_{t+1}}(A_{t+1})\cdots E(k_s,j_s-1)S_{i_s,j_s}(A_s)f_{\mu,\lambda}.$$

Now the required formula follows similarly to (iv). □

For $1\leqslant i<j\leqslant n$ and $A\subset(i..j)$, we define the polynomial $\mathcal{K}_{i,j}(A)$ of $\mathbb{Z}[x_i,\ldots,x_{j-1},y_{i+1},\ldots,y_j]$ as in [1, 4.12] by the formula

$$\mathcal{K}_{i,j}(A) := \sum_{B\subset(i..j)}\left(\mathcal{H}_{i,j}(A,B)\prod_{t\in B\cup\{i\}}(y_{t+1}-x_t)\right).$$

We define $H^{\mu}_{i,j}(A,B)$ by evaluating $\mathcal{H}_{i,j}(A,B)$ at $x_q := \mathrm{res}_p(q,\mu_q)$ and define $K^{\mu,\lambda,k}_{i,j}(A)$ by evaluating $\mathcal{K}_{i,j}(A)$ at

$$\begin{array}{ll} x_q := \mathrm{res}_p(q,\mu_q) & \text{for } 1\leqslant q<n,\\ y_q := \mathrm{res}_p(q,\lambda_q+1) & \text{for } 1<q\leqslant k,\\ y_q := \mathrm{res}_p(q,\mu_q+1) & \text{for } k<q<n, \end{array} \tag{1}$$

where $1+\delta_{j=n}(n-1)\leqslant k\leqslant n$. For $1\leqslant i\leqslant t<n$ and $1+\delta_{t+1=n}(n-1)\leqslant k\leqslant n$, let $B^{\mu,\lambda,k}(i,t)$ denote the element of $\mathbb{Z}/p\mathbb{Z}$ obtained from $y_{t+1}-x_i$ by substitution (1). We also abbreviate $K^{\mu,\lambda}_{i,j}(A) := K^{\mu,\lambda,n}_{i,j}(A)$ and $B^{\mu,\lambda}(i,t) := B^{\mu,\lambda,n}(i,t)$.

Remark 1. Clearly $B^{\mu,\lambda,k}(i,t) = t-i+\mu_i-\mu_{t+1}$ for $k\leqslant t$ and $B^{\mu,\lambda,k}(i,t) = t-i+\mu_i-\lambda_{t+1}$ for $k>t$. In particular, $B^{\mu,\lambda,k}(i,t) = B^{\mu,\lambda,i}(i,t)$ for $k\leqslant i$ and $B^{\mu,\lambda,k}(i,t) = B^{\mu,\lambda,t+1}(i,t)$ for $k>t$.

The next result is actually proved in [4, Proposition 4.5]. Recall that we have defined $a_t = \sum_{j=1}^{t}(\lambda_j-\mu_j)$.

Proposition 3. *Given integers* $1 \leqslant d_1 < d_1' \leqslant d_2 < d_2' \leqslant \cdots \leqslant d_r < d_r' \leqslant n$, *we have*

$$\left(\prod_{t=1}^{n-1} E_t^{(a_t+\delta_{t\in G})}\right) F_{d_1,d_1'} \cdots F_{d_r,d_r'} f_{\mu,\lambda} = \prod_{q=1}^{r} (\mu_{d_q} - \lambda_{d_q+1}) f_\lambda,$$

where $G = [d_1..d_1') \cup \cdots \cup [d_r..d_r')$.

Lemma 4. *Under the hypothesis of Lemma 2, we have*

$$\left(\prod_{t=1}^{n-1} E_t^{(a_t+\delta_{t\in G})}\right) v = K_{i_1,j_1}^{\mu,\lambda,k_1}(A_1) \cdots K_{i_s,j_s}^{\mu,\lambda,k_s}(A_s) f_\lambda,$$

where $G = [i_1..k_1) \cup \cdots \cup [i_s..k_s)$.

Proof. By Lemma 1, we have $E(k_t, j_t{-}1)F_{i_t,j_t}^B \equiv F_{i_t,k_t}^{B_{i_t..k_t}} \prod_{q\in B\cup\{i_t\}, q\geqslant k_t}(H_q - H_{q+1})$ modulo the left ideal of $U(n)$ generated by $E_{k_t}, \ldots, E_{j_t-1}$. Thus taking into account [1, Remark 4.8], we get

$$v = \prod_{t=1}^{s} \sum_{B_t \subset (i_t..j_t)} \left(H_{i_t,j_t}^{\mu}(A_t, B_t) F_{i_t,k_t}^{(B_t)_{i_t..k_t}} \prod_{\substack{q\in B_t\cup\{i_t\}\\ q\geqslant k_t}} (\mu_q - \mu_{q+1}) \right) f_{\mu,\lambda}. \quad (2)$$

By Proposition 3, we have

$$\left(\prod_{t=1}^{n-1} E_t^{(a_t+\delta_{t\in G})}\right) F_{i_1,k_1}^{(B_1)_{i_1..k_1}} \cdots F_{i_s,k_s}^{(B_s)_{i_s..k_s}} f_{\mu,\lambda} = \prod_{t=1}^{s} \prod_{\substack{q\in B_t\cup\{i_t\}\\ q<k_t}} (\mu_q - \lambda_{q+1}) f_\lambda.$$

Substituting this into (2) completes the proof. □

Let V_n be the set of all sequences $x = \big((i_1, k_1, j_1, A_1), \ldots, (i_s, k_s, j_s, A_s)\big)$ such that

$$1 \leqslant i_1 < j_1 < \cdots < i_s < j_s \leqslant n; \quad A_1 \subset (i_1..j_1), \ldots, A_s \subset (i_s..j_s);$$
$$i_1 \leqslant k_1 \leqslant j_1, \ldots, i_s \leqslant k_s \leqslant j_s; \quad j_s = n \text{ implies } k_s = n.$$

Moreover, we put $\Phi(x) := E(k_1, j_1{-}1)S_{i_1,j_1}(A_1) \cdots E(k_s, j_s{-}1)S_{i_s,j_s}(A_s)$ and $K^{\mu,\lambda}(x) := K_{i_1,j_1}^{\mu,\lambda,k_1}(A_1) \cdots K_{i_s,j_s}^{\mu,\lambda,k_s}(A_s)$. In what follows, we assume that the product of two finite sequences $a = (a_1, \ldots, a_s)$ and $b = (b_1, \ldots, b_t)$ equals $ab = (a_1, \ldots, a_s, b_1, \ldots, b_t)$.

Let $x, x' \in V_n$. We write $x \xrightarrow{l} x'$ if there exists a representation $x = a\big((i,k,j,A)\big)b$ such that one of the following conditions holds:

- $x' = a\big((i, k-1, j, A)\big)b$, $l = k-1$, $i < k < n$;
- $x' = a\big((i+1, k, j, A)\big)b$, $l = i$, $i+1 \notin A$, $i < k-1$;
- $x' = a\big((i, l, l, A_{i..l}), (l{+}1, k, j, A_{l+1..j})\big)b$, $l \in (i..k{-}1)$, $l \in A$, $l{+}1 \notin A$.

The above definitions are made exactly to ensure the following property.

Lemma 5. *Let* $x, x' \in V_n$. *If* $x \xrightarrow{l} x'$ *then* $E_l\Phi(x)f_{\mu,\lambda} = \pm\Phi(x')f_{\mu,\lambda}$.

Proof follows directly from Lemma 2(iv),(v). □

We say that x' *follows* from x if there are $x_0, \ldots, x_m \in V_n$ and integers $l_0, \ldots, l_{m-1}$ such that $x = x_0$, $x' = x_m$ and $x_t \xrightarrow{l_t} x_{t+1}$ for $0 \leqslant t < m$. In particular, every element of V_n follows from itself.

Theorem 6. *Let $x \in V_n$. The equality $\Phi(x)f_{\mu,\lambda} = 0$ holds if and only if $K^{\mu,\lambda}(x') = 0$ for any x' following from x.*

Proof. It follows from Lemmas 5 and 4 that $\Phi(x)f_{\mu,\lambda} = 0$ implies $K^{\mu,\lambda}(x') = 0$ for any x' following from x.

Let $x = \big((i_1, k_1, j_1, A_1), \dots, (i_s, k_s, j_s, A_s)\big)$. We prove the reverse implication by induction on $\sum_{t=1}^{s}(k_t - i_t)$. The induction starts by noting that this sum is always non-negative. So we suppose that the reverse implication is true for smaller values of this sum. By Lemma 2(ii),(iii), we get $E_l^{(m)}\Phi(x)f_{\mu,\lambda} = 0$ if $l < n-1$ and $m > 1$ or if $m \geqslant 1$ and $l \in [1..n-1) \setminus \big([i_1..k_1) \cup \dots \cup [i_s..k_s)\big)$.

However $E_l\Phi(x)f_{\mu,\lambda} = 0$ also for $l \in [1..n-1) \cap \big([i_1..k_1) \cup \dots \cup [i_s..k_s)\big)$ by Lemma 5 and the inductive hypothesis. Thus $\Phi(x)f_{\mu,\lambda}$ is a $U(n-1)$-high weight vector of weight $\nu = \mu - \sum_{t=1}^{s}(\varepsilon_{i_t} - \varepsilon_{k_t})$, where $\varepsilon_i = (0^{i-1}, 1, 0^{n-1-i})$ for $i < n$ and $\varepsilon_n = (0^{n-1})$. It follows from [4, Corollary 3.3] that $\Phi(x)f_{\mu,\lambda} = 0$ if $\nu \longleftarrow \lambda$ does not hold and that $\Phi(x)f_{\mu,\lambda} = cf_{\nu,\lambda}$ for some $c \in K$ if $\nu \longleftarrow \lambda$. We need to consider only the latter case. By the last equation of the introduction and Lemma 4, we have $cf_\lambda = X(cf_{\nu,\lambda}) = X\Phi(x)f_{\mu,\lambda} = K^{\mu,\lambda}(x)f_\lambda = 0$, where $X = \prod_{t=1}^{n-1} E_t^{(a_t + \delta_{t\in G})}$ and $G = [i_1..k_1) \cup \dots \cup [i_s..k_s)$. Hence $c = 0$ and $\Phi(x)f_{\mu,\lambda} = 0$. $\square$

The next corollary follows from Theorem 6 and the following simple fact: if $x \in V_n$ and $x = x_1x_2$ then x' follows from x if and only if there are sequences x_1' and x_2' following from x_1 and x_2 respectively such that $x' = x_1'x_2'$.

Corollary 7. *Let $x \in V_n$ and $x = x_1x_2$. Then $\Phi(x)f_{\mu,\lambda} = 0$ if and only if $\Phi(x_1)f_{\mu,\lambda} = 0$ or $\Phi(x_2)f_{\mu,\lambda} = 0$.*

3. Removing one node

We say that a map $\theta : A \to \mathbb{Z}$, where $A \subset \mathbb{Z}$, is *weakly increasing* (*weakly decreasing*) if $\theta(a) \geqslant a$ (resp. $\theta(a) \leqslant a$) for any $a \in A$. We need the following facts about the polynomials $\mathcal{K}_{i,j}(A)$.

Proposition 8. *Let $1 \leqslant i < j \leqslant n$, $1 + \delta_{j=n}(n-1) \leqslant k \leqslant n$, $A \subset (i..j)$ and there exists a weakly increasing injection $\theta : (i..j) \setminus A \to (i..j)$ such that $B^{\mu,\lambda,k}(t, \theta(t)) = 0$ for any $t \in (i..j) \setminus A$. Then*

$$K_{i,j}^{\mu,\lambda,k}(A) = \prod_{t \in [i..j) \setminus \operatorname{Im}\theta} B^{\mu,\lambda,k}(i,t).$$

Proof. The result is obtained from [4, Lemma 4.4] by substitution (1). $\square$

Lemma 9. *For $i < j-1$ and $A \subset (i..j)$, we have*

(i) $\mathcal{K}_{i,j}(A) = \mathcal{K}_{i,j-1}(A)$ *if $j - 1 \notin A$;*

(ii) $\mathcal{K}_{i,j}(A) = \mathcal{K}_{i,j-1}(A \setminus \{j-1\})(y_j - x_k) + \delta_{k\neq i}\mathcal{K}_{i,j-1}(\{k\} \cup A \setminus \{j-1\})$, *where $k = \max[i..j) \setminus A$, if $j - 1 \in A$.*

Proof. We put $\bar{A} = (i..j) \setminus A$. In this proof, we use [1, Lemma 4.13(i)] for a self-contained form of $\mathcal{K}_{i,j}(A)$ and the following notation of [1]: if $D \subset (i..j)$ and $k > i$ then $D_i(k) = \max\{t \in D \cup \{i\} : t < k\}$.

(i) If $D \subset \bar{A} \setminus \{j-1\}$ then $(D \cup \{j-1\})_i(t) = D_i(t)$ for $t < j$, $(D \cup \{j-1\})_i(j) = j-1$ and $D_i(j) = D_i(j-1)$. Hence we get

$$\mathcal{K}_{i,j}(A) = \sum_{D \subset \bar{A} \setminus \{j-1\}} (-1)^{|D|} \left(\frac{\prod\limits_{t \in (i..j]} (y_t - x_{D_i(t)})}{\prod\limits_{t \in \bar{A}} (x_t - x_{D_i(t)})} - \frac{\prod\limits_{t \in (i..j]} (y_t - x_{(D \cup \{j-1\})_i(t)})}{\prod\limits_{t \in \bar{A}} (x_t - x_{(D \cup \{j-1\})_i(t)})} \right) =$$

$$\sum_{D \subset \bar{A} \setminus \{j-1\}} (-1)^{|D|} \left(\frac{\prod\limits_{t \in (i..j-1]} (y_t - x_{D_i(t)})}{\prod\limits_{t \in \bar{A} \setminus \{j-1\}} (x_t - x_{D_i(t)})} \frac{(y_j - x_{D_i(j-1)}) - (y_j - x_{j-1})}{x_{j-1} - x_{D_i(j-1)}} \right) = \mathcal{K}_{i,j-1}(A).$$

(ii) If $k = i$ then $A = (i..j)$, $\mathcal{K}_{i,j}(A) = \prod_{t \in (i..j]} (y_t - x_i)$, $\mathcal{K}_{i,j-1}(A \setminus \{j-1\}) = \prod_{t \in (i..j-1]} (y_t - x_i)$ (by part (i)) and the required formula follows.

Therefore, we consider the case $k \neq i$. We have

$$\mathcal{K}_{i,j}(A) = (y_j - x_k) \sum_{D \subset \bar{A}} (-1)^{|D|} \frac{\prod_{t \in (i..j-1]} (y_t - x_{D_i(t)})}{\prod_{t \in \bar{A}} (x_t - x_{D_i(t)})}$$
$$+ \sum_{D \subset \bar{A}} (-1)^{|D|} (x_k - x_{D_i(j)}) \frac{\prod_{t \in (i..j-1]} (y_t - x_{D_i(t)})}{\prod_{t \in \bar{A}} (x_t - x_{D_i(t)})}.$$

Part (i) shows that the first sum equals $\mathcal{K}_{i,j}(A \setminus \{j-1\})$. Let us look at the second sum. If $k \in D$ then $D_i(j) = k$ and the summands corresponding to such sets D can be omitted. If $k \notin D$ then $D_i(j) = D_i(k)$ and this summand equals

$$(-1)^{|D|} \frac{\prod_{t \in (i..j-1]} (y_t - x_{D_i(t)})}{\prod_{t \in \bar{A} \setminus \{k\}} (x_t - x_{D_i(t)})}.$$

Thus the second sum equals $\mathcal{K}_{i,j-1}(\{k\} \cup A \setminus \{j-1\})$. $\square$

Next, we are going to prove the result similar to [5, Proposition 3.2], where we replace the $U(n)$-high weight vector v_+ by the $U(n-1)$-high weight vector $f_{\mu,\lambda}$. The general scheme of proof is borrowed from [5, Proposition 3.2], although some changes are necessary. We shall use Theorem 6 and Lemma 9 to make them. In what follows, we say that a formula $M = [b_1..c_1] \cup \dots \cup [b_N..c_N]$ is *the decomposition of M into the union of connected components* if $b_i \leqslant c_i$ for $1 \leqslant i \leqslant N$ and $c_i < b_{i+1} - 1$ for $1 \leqslant i < N$.

Definition 10. *Let $1 \leqslant i < j \leqslant n$, $M \subset (i..j)$ and $M = [b_1..c_1] \cup \dots \cup [b_N..c_N]$ be the decomposition of M into the union of connected components. We say that M satisfies the condition $\pi^{\mu,\lambda}_{i,j}(v)$ if $1 \leqslant v \leqslant N+1$ and for any $k = 1 + \delta_{b_v - 1 = n}(n-1), \dots, n$ there exists a weakly increasing injection $\theta_k : \{i\} \cup [b_1..c_1] \cup \dots \cup [b_{v-1}..c_{v-1}] \to [i..b_v - 1)$ such that $B^{\mu,\lambda,k}(x, \theta_k(x)) = 0$ for any admissible x, where we assume $b_{N+1} = j+1$.*

Lemma 11. *Let $1 \leqslant i < j \leqslant n$ and $A \subset (i..j)$ be such that $(i..j) \setminus A$ satisfies $\pi^{\mu,\lambda}_{i,j}(v)$ for some v. Then $K^{\mu,\lambda,k}_{i,j}(A) = 0$ for $1 + \delta_{j=n}(n-1) \leqslant k \leqslant n$.*

Proof. Let $(i..j) \setminus A = [b_1..c_1] \cup \dots \cup [b_N..c_N]$ be the decomposition into the union of connected components. Note that if $v = N+1$, then the required equalities immediately follow from Proposition 8.

Indeed, take any $k = 1 + \delta_{j=n}(n-1), \dots, n$. Since in this case $b_v - 1 = j$, Definition 10 ensures that there exists a weakly increasing injection θ_k :

$\{i\} \cup ((i..j) \setminus A) \to [i..j)$ such that $B^{\mu,\lambda,k}(x, \theta_k(x)) = 0$ for any admissible x. Taking the restriction of θ_k to $(i..j) \setminus A$ for θ in Proposition 8, we obtain

$$K_{i,j}^{\mu,\lambda,k}(A) = \prod_{t \in [i..j) \setminus \operatorname{Im}\theta} B^{\mu,\lambda,k}(i,t).$$

The last product equals zero, since $B^{\mu,\lambda,k}(i, \theta_k(i)) = 0$ and $\theta_k(i) \in [i..j) \setminus \operatorname{Im}\theta$.

Let us prove the lemma by induction on $j - i$. The case $j - i = 1$ follows from the above remark. Now let $v \leqslant N$, $j - i > 1$ and suppose that the lemma is true for smaller values of this difference. Take any $k = 1 + \delta_{j=n}(n-1), \dots, n$. By Lemma 9, we have

$$K_{i,j}^{\mu,\lambda,k}(A) = K_{i,j-1}^{\mu,\lambda,k}(A \setminus \{j-1\})B + K_{i,j-1}^{\mu,\lambda,k}(\{c_N\} \cup A \setminus \{j-1\})$$

if $c_N < j - 1$ and

$$K_{i,j}^{\mu,\lambda,k}(A) = K_{i,j-1}^{\mu,\lambda,k}(A)$$

if $c_N = j - 1$, where B is the element of $\mathbb{Z}/p\mathbb{Z}$ obtained from $y_j - x_{c_N}$ by substitution (1). Clearly, the sets $(i..j-1) \setminus (A \setminus \{j-1\})$ and $(i..j-1) \setminus (\{c_N\} \cup A \setminus \{j-1\})$ in the former case and the set $(i..j-1) \setminus A$ in the latter case satisfy the condition $\pi_{i,j-1}^{\mu,\lambda}(v)$. $\square$

Theorem 12. *Let $1 \leqslant i < j \leqslant n$ and $A \subset (i..j)$. Then $S_{i,j}(A)f_{\mu,\lambda} = 0$ if and only if $(i..j) \setminus A$ satisfies $\pi_{i,j}^{\mu,\lambda}(v)$ for some v.*

Proof. Let $\bar{A} = (i..j) \setminus A$ and $\bar{A} = [b_1..c_1] \cup \dots \cup [b_N..c_N]$ be the decomposition into the union of connected components. We put $x_k = \big((i,k,j,A)\big)$ for brevity. It should be kept in mind that $\Phi(x_j) = S_{i,j}(A)$.

We prove the theorem by induction on $|\bar{A}|$. Suppose $\bar{A} = \varnothing$. Then all the sequences following from x_j are x_k, where $i + \delta_{j=n}(j-i) \leqslant k \leqslant j$. By Theorem 6, $\Phi(x_j)f_{\mu,\lambda} = 0$ if and only if $K^{\mu,\lambda}(x_k) = 0$ for any $k = i + \delta_{j=n}(j-i), \dots, j$. Applying Proposition 8, we see that $\Phi(x_j)f_{\mu,\lambda} = 0$ if and only if for any $k = i + \delta_{j=n}(j-i), \dots, j$ there is $t_k \in [i..j)$ such that $B^{\mu,\lambda,k}(i, t_k) = 0$. In view of Remark 1, this assertion is equivalent to $\pi_{i,j}^{\mu,\lambda}(1)$.

Now suppose that $\bar{A} \neq \varnothing$ and that the theorem holds for smaller values of $|\bar{A}|$.

"If part". By [1, 4.11(ii)] for any $m = 1, \dots, N$, we have $E_{b_m-1}S_{i,j}(A)f_{\mu,\lambda} = -S_{i,b_m-1}(A_{i..b_m-1})\ S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda}$. Note that

$$\begin{aligned} A_{i..b_m-1} &= (i..b_m - 1) \setminus \big([b_1..c_1] \cup \dots \cup [b_{m-1}..c_{m-1}]\big), \\ A_{b_m..j} &= (b_m..j) \setminus \big((b_m..c_m] \cup \dots \cup [b_N..c_N]\big). \end{aligned} \tag{3}$$

If $m \leqslant v - 1$ then $(b_m..c_m] \cup \dots \cup [b_N..c_N]$ satisfies $\pi_{b_m,j}^{\mu,\lambda}(v - m + 1 - \delta_{b_m=c_m})$, whence by the inductive hypothesis $S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda} = 0$. If $m \geqslant v$ then $i < b_m - 1$ and $[b_1..c_1] \cup \dots \cup [b_{m-1}..c_{m-1}]$ satisfies $\pi_{i,b_m-1}^{\mu,\lambda}(v)$, whence by the inductive hypothesis $S_{i,b_m-1}(A_{i..b_m-1})f_{\mu,\lambda} = 0$. Since the elements $S_{i,b_m-1}(A_{i..b_m-1})$ and $S_{b_m,j}(A_{b_m..j})$ commute, we have in both cases

$$E_{b_m-1}S_{i,j}(A)f_{\mu,\lambda} = 0. \tag{4}$$

Let us prove by induction on $s = 0, \dots, j-i$ that in the case $j < n$ the conditions

$$K_{i,j}^{\mu,\lambda,j}(A) = 0, \quad \dots, \quad K_{i,j}^{\mu,\lambda,j-s+1}(A) = 0, \quad \Phi(x_{j-s})f_{\mu,\lambda} = 0 \qquad (5)$$

imply $\Phi(x_j)f_{\mu,\lambda} = 0$. It is obviously true for $s = 0$. Suppose that $0 < s \leqslant j-i$, conditions (5) hold and the assertion is true for smaller values of s. By the inductive hypothesis it suffices to prove that $\Phi(x_{j-s+1})f_{\mu,\lambda} = 0$. Let $x_{j-s+1} \xrightarrow{l} x'$. We have either $x' = x_{j-s}$ or $l = b_m - 1 < j - s$. Since in the former case $\Phi(x')f_{\mu,\lambda} = 0$ by (5), we shall consider the latter case. We have

$$\Phi(x')f_{\mu,\lambda} = E_{b_m-1}\Phi(x_{j-s+1})f_{\mu,\lambda} = E_{b_m-1}E(j-s+1, j-1)S_{i,j}(A)f_{\mu,\lambda}$$
$$= E(j-s+1, j-1)E_{b_m-1}S_{i,j}(A)f_{\mu,\lambda} = 0.$$

To obtain the last equality, we used (4). Since $K^{\mu,\lambda}(x_{j-s+1}) = K_{i,j}^{\mu,\lambda,j-s+1}(A) = 0$, we get $\Phi(x_{j-s+1})f_{\mu,\lambda} = 0$ by Theorem 6.

Note that nothing follows from x_i except itself. Therefore, applying the above assertion for $s = j-i$ and Theorem 6, we see that to prove $\Phi(x_j)f_{\mu,\lambda} = 0$ in the case $j < n$, it suffices to prove $K_{i,j}^{\mu,\lambda,k}(A) = 0$ for $i \leqslant k \leqslant j$. The last equalities follow from Lemma 11.

If $j = n$ then $x_j \xrightarrow{l} x'$ holds if and only if $l = b_m - 1$, where $1 \leqslant m \leqslant N$. In that case $\Phi(x')f_{\mu,\lambda} = 0$ by (4). Therefore, applying Theorem 6, we see that to prove $\Phi(x_j)f_{\mu,\lambda} = 0$ in the case $j = n$, it suffices to prove $K_{i,j}^{\mu,\lambda}(A) = 0$. The last equality follows from Lemma 11.

"Only if part". Suppose $\bar{A}$ satisfies the condition $\pi_{i,j}^{\mu,\lambda}(v)$ for no v. Multiplying the equality $\Phi(x_j)f_{\mu,\lambda} = 0$ by E_{b_m-1}, where $1 \leqslant m \leqslant N$, we get $S_{i,b_m-1}(A_{i..b_m-1})S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda} = 0$ according to [1, 4.11(ii)]. By Corollary 7, either $S_{i,b_m-1}(A_{i..b_m-1})f_{\mu,\lambda} = 0$ or $S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda} = 0$. The former case is impossible since the inductive hypothesis would yield that $(i..b_m-1) \setminus A_{i..b_m-1}$ satisfies $\pi_{i,b_m-1}^{\mu,\lambda}(v)$ for some $v \leqslant m$ (see (3)). But then $\bar{A}$ would satisfy $\pi_{i,j}^{\mu,\lambda}(v)$, which is wrong. Therefore $S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda} = 0$ for any $m = 1, \dots, N$.

We shall use this fact to prove by downward induction on $u = 1, \dots, N+1$ the following property:

for any $k = 1 + \delta_{j=n}(n-1), \dots, n$, there is a weakly increasing injection $d_k : [b_u..c_u] \cup \dots \cup [b_N..c_N] \to (i..j)$ such that $B^{\mu,\lambda,k}(x, d_k(x)) = 0$ for any admissible x. (6)

This is obviously true for $u = N+1$. Therefore, we suppose that $1 \leqslant u \leqslant N$ and property (6) is proved for greater u. Fix an arbitrary $k = 1 + \delta_{j=n}(n-1), \dots, n$. Since $S_{b_u,j}(A_{b_u..j})f_{\mu,\lambda} = 0$, the inductive hypothesis asserting that the current lemma is true for smaller values of $|\bar{A}|$ implies that $(b_u..j) \setminus A_{b_u..j}$ satisfies $\pi_{b_u,j}^{\mu,\lambda}(v)$ for some v. As a consequence, there is a weakly increasing injection $e_k : [b_u..c_u] \cup \dots \cup [b_{u+w-1}..c_{u+w-1}] \to [b_u..b_{u+w}-1)$ such that $B^{\mu,\lambda,k}(x, d_k(x)) = 0$ for any admissible x (here $w = v - 1 + \delta_{b_u=c_u}$ and $b_{N+1} = j+1$). The inductive hypothesis asserting that property (6) holds for

$u+w$ allows us to extend e_k to the required injection d_k. Thus property (6) is proved.

Take any $k = i+\delta_{j=n}(j-i), \dots, j$. Applying property (6) for $u=1$, the fact that x_k follows from x_j, and Proposition 8, we get

$$0 = K^{\mu,\lambda}(x_k) = K^{\mu,\lambda,k}_{i,j}(A) = \prod_{t\in[i..j)\setminus \operatorname{Im} d_k} B^{\mu,\lambda,k}(i,t).$$

Therefore, there is $t' \in [i..j) \setminus \operatorname{Im} d_k$ such that $B^{\mu,\lambda,k}(i,t') = 0$. Putting $\theta_k(t) = d_k(t)$ for $t \in [b_1..c_1] \cup \dots \cup [b_N..c_N]$ and $\theta_k(i) = t'$, we get a map required in Definition 10. This fact together with Remark 1 shows that $\bar{A}$ satisfies $\pi^{\mu,\lambda}_{i,j}(N+1)$, contrary to assumption. □

Following [4], we introduce the following sets:

$$\begin{aligned} \mathfrak{C}^{\mu}(i,j) &:= \{a : i < a < j, C^{\mu}(i,a) = 0\}, \\ \mathfrak{B}^{\mu,\lambda}(i,j) &:= \{a : i \leqslant a < j, B^{\mu,\lambda}(i,a) = 0\}, \end{aligned}$$

where $C^{\mu}(i,a)$ is the residue class of $a - i + \mu_i - \mu_a$ modulo p as in [4].

Theorem 13. *Let $1 \leqslant i < n$.*

(i) *Let $A \subset (i..n)$. Then $S_{i,n}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector if and only if there is a weakly increasing injection $d : (i..n) \setminus A \to (i..n)$ such that $B^{\mu,\lambda}(x, d(x)) = 0$ for any admissible x and $B^{\mu,\lambda}(i,t) \neq 0$ for any $t \in [i..n) \setminus \operatorname{Im} d$.*

(ii) *There is some $A \subset (i..n)$ such that $S_{i,n}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector if and only if there is a weakly decreasing injection from $\mathfrak{B}^{\mu,\lambda}(i,n)$ to $\mathfrak{C}^{\mu}(i,n)$.*

Proof. (i) It is clear from [1, 4.11(ii)], Theorem 12 and Proposition 8 that $S_{i,n}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector for such A. Conversely, if $S_{i,n}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector then, arguing as in the "only if part" of Theorem 12, we get that there is a weakly increasing injection $d : (i..n)\setminus A \to (i..n)$ such that $B^{\mu,\lambda}(x, d(x)) = 0$ for any admissible x. Now by Proposition 8, we have $0 \neq K^{\mu,\lambda}(i,n)(A) = \prod_{t\in[i..n)\setminus \operatorname{Im} d} B^{\mu,\lambda}(i,t)$.

(ii) If ε is such an injection, then it suffices to put $A = (i..n) \setminus \operatorname{Im}\varepsilon$, take for d the inverse map of ε and apply part (i). Conversely, let $S_{i,n}(A)f_{\mu,\lambda}$ be a non-zero $U(n-1)$-high weight vector for some $A \subset (i..n)$ and let d be an injection, whose existence is claimed by part (i). Now the result follows from the following two observations: $\mathfrak{B}^{\mu,\lambda}(i,n) \subset \operatorname{Im} d$; $d(x) \in \mathfrak{B}^{\mu,\lambda}(i,n)$ implies $x \in \mathfrak{C}^{\mu}(i,n)$. □

Remark 2. If we obtain a non-zero $U(n-1)$-high weight vector in Theorem 13, then it is a scalar multiple of $f_{\nu,\lambda}$, where $\nu = \mu - \varepsilon_i$ and $\varepsilon_i = (0^{i-1}, 1, 0^{n-1-i})$.

4. Moving one node

Definition 14. *Let $1 \leqslant i < j-1 < n-1$, $M \subset (i..j-1)$ and $M = [b_1..c_1] \cup \dots \cup [b_N..c_N]$ be the decomposition of M into the union of connected components. We say that M satisfies the condition $\bar{\pi}^{\mu,\lambda}_{i,j}(v)$ if $1 \leqslant v \leqslant N+1$ and for any $k = 1, \dots, j-1$ there exists a weakly increasing injection θ_k :*

$\{i\}\cup[b_1..c_1]\cup\cdots\cup[b_{v-1}..c_{v-1}]\to[i..b_v-1)$ *such that* $B^{\mu,\lambda,k}(x,\theta_k(x))=0$ *for any admissible* x, *where we assume* $b_{N+1}=j+1$.

Remark 3. If in the above definition for some $k=1,\dots,j-1$, the inequality $\theta_k(x)<k$ holds for any admissible x, then the maps θ_l : $\{i\}\cup[b_1..c_1]\cup\cdots\cup[b_{v-1}..c_{v-1}]\to[i..b_v-1)$ for $k<l\leqslant n$ such that $B^{\mu,\lambda,l}(x,\theta_l(x))=0$ for any admissible x, can be defined equal to θ_k.

Indeed, it follows from Remark 1 that for $k<l\leqslant n$ we have $B^{\mu,\lambda,l}(x,\theta_k(x))=B^{\mu,\lambda,k}(x,\theta_k(x))=0$ for any admissible x. In particular (taking $k=j-1$), we obtain that for $v\leqslant N$ the set M (that consists of N connected components) satisfies the condition $\bar\pi^{\mu,\lambda}_{i,j}(v)$ if and only if it satisfies the condition $\pi^{\mu,\lambda}_{i,j}(v)$.

Theorem 15. *Let* $1\leqslant i<j-1<n-1$ *and* $A\subset(i..j)$ *such that* $j-1\in A$. *Then* $E_{j-1}S_{i,j}(A)f_{\mu,\lambda}=0$ *if and only if* $(i..j-1)\setminus A$ *satisfies* $\bar\pi^{\mu,\lambda}_{i,j}(v)$ *for some* v.

Proof. Let $\bar A=(i..j)\setminus A$ and $\bar A=[b_1..c_1]\cup\cdots\cup[b_N..c_N]$ be the decomposition into the union of connected components. We put $x_k=\big((i,k,j,A)\big)$ for brevity.

We prove the theorem by induction on $|\bar A|$. Suppose $\bar A=\varnothing$. Then all the sequences following from x_{j-1} are x_k, where $i\leqslant k\leqslant j-1$. By Theorem 6, $\Phi(x_{j-1})f_{\mu,\lambda}=0$ if and only if $K^{\mu,\lambda}(x_k)=0$ for any $k=i,\dots,j-1$. Applying Proposition 8, we see that $\Phi(x_{j-1})f_{\mu,\lambda}=0$ if and only if for any $k=i,\dots,j-1$ there is $t_k\in[i..j)$ such that $B^{\mu,\lambda,k}(i,t_k)=0$. In view of Remark 1, this assertion is equivalent to $\bar\pi^{\mu,\lambda}_{i,j}(1)$.

Now suppose that $\bar A\neq\varnothing$ and that the theorem holds for smaller values of $|\bar A|$.

"If part". If $v\leqslant N$ then $\bar A$ satisfies $\pi^{\mu,\lambda}_{i,j}(v)$ by Remark 3. Hence by Theorem 12, we have $S_{i,j}(A)f_{\mu,\lambda}=0$ and the desired result follows.

So we shall consider the case $v=N+1$. For any $m=1,\dots,N$, the elements E_{b_m-1} and E_{j-1} commute and by [1, 4.11(ii)] we have $E_{b_m-1}E_{j-1}$ $S_{i,j}(A)f_{\mu,\lambda}=-S_{i,b_m-1}(A_{i..b_m-1})\ E_{j-1}S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda}$. Note that

$$\begin{aligned}A_{i..b_m-1}&=(i..b_m-1)\setminus\big([b_1..c_1]\cup\cdots\cup[b_{m-1}..c_{m-1}]\big),\\ A_{b_m..j}&=(b_m..j)\setminus\big((b_m..c_m]\cup\cdots\cup[b_N..c_N]\big).\end{aligned}\tag{7}$$

Obviuosly, the set $(b_m..c_m]\cup\cdots\cup[b_N..c_N]$ satisfies $\bar\pi^{\mu,\lambda}_{b_m,j}(N+2-m-\delta_{b_m=c_m})$, whence by the inductive hypothesis $E_{j-1}S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda}=0$. Thus we have

$$E_{b_m-1}E_{j-1}S_{i,j}(A)f_{\mu,\lambda}=0.\tag{8}$$

Let us prove by induction on $s=0,\dots,j-i-1$ that the conditions

$$K^{\mu,\lambda,j-1}_{i,j}(A)=0,\quad\dots,\quad K^{\mu,\lambda,j-s}_{i,j}(A)=0,\quad\Phi(x_{j-1-s})f_{\mu,\lambda}=0\tag{9}$$

imply $\Phi(x_{j-1})f_{\mu,\lambda}=0$. It is obviously true for $s=0$. Suppose that $0<s\leqslant j-i-1$, conditions (9) hold and the assertion is true for smaller values of s. By the inductive hypothesis it suffices to prove that $\Phi(x_{j-s})f_{\mu,\lambda}=0$. Let $x_{j-s}\xrightarrow{l}x'$. We have either $x'=x_{j-s-1}$ or $l=b_m-1<j-s-1$. Since in

the former case $\Phi(x')f_{\mu,\lambda}=0$ by (9), we shall consider the latter case. We have

$$\Phi(x')f_{\mu,\lambda}=E_{b_m-1}\Phi(x_{j-s})f_{\mu,\lambda}=E_{b_m-1}E(j-s,j-2)E_{j-1}S_{i,j}(A)f_{\mu,\lambda}$$
$$=E(j-s,j-2)E_{b_m-1}E_{j-1}S_{i,j}(A)f_{\mu,\lambda}=0.$$

To obtain the last equality, we used (8). Since $K^{\mu,\lambda}(x_{j-s})=K^{\mu,\lambda,j-s}_{i,j}(A)=0$, we get $\Phi(x_{j-s})f_{\mu,\lambda}=0$ by Theorem 6.

Note that nothing follows from x_i except itself. Therefore, applying the above assertion for $s=j-i-1$ and Theorem 6, we see that to prove $\Phi(x_{j-1})f_{\mu,\lambda}=0$, it suffices to prove $K^{\mu,\lambda,k}_{i,j}(A)=0$ for $i\leqslant k\leqslant j-1$. The last equalities follow from Proposition 8.

"Only if part". Suppose $\bar{A}$ satisfies the condition $\bar{\pi}^{\mu,\lambda}_{i,j}(v)$ for no v. Multiplying the equality $\Phi(x_{j-1})f_{\mu,\lambda}=0$ by E_{b_m-1}, where $1\leqslant m\leqslant N$, we get $S_{i,b_m-1}(A_{i..b_m-1})E_{j-1}S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda}=0$ according to [1, 4.11(ii)]. By Corollary 7, either $S_{i,b_m-1}(A_{i..b_m-1})f_{\mu,\lambda}=0$ or $E_{j-1}S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda}=0$. The former case is impossible since Theorem 12 would yield that $(i..b_m-1)\setminus A_{i..b_m-1}$ satisfies $\pi^{\mu,\lambda}_{i,b_m-1}(v)$ for some $v\leqslant m$ (see (3)). But then $\bar{A}$ would satisfy $\pi^{\mu,\lambda}_{i,j}(v)$ and thus also would satisfy $\bar{\pi}^{\mu,\lambda}_{i,j}(v)$, which is wrong. Therefore $E_{j-1}S_{b_m,j}(A_{b_m..j})f_{\mu,\lambda}=0$ for any $m=1,\dots,N$.

We shall use this fact to prove by downward induction on $u=1,\dots,N+1$ the following property:

$$\text{for any } k=1,\dots,j-1, \text{ there is a weakly increasing injection } d_k:[b_u..c_u]\cup\dots\cup[b_N..c_N]\to(i..j) \text{ such that } B^{\mu,\lambda,k}(x,d_k(x))=0 \text{ for any admissible } x. \tag{10}$$

This is obviously true for $u=N+1$. Therefore, we suppose that $1\leqslant u\leqslant N$ and property (10) is proved for greater u. Fix an arbitrary $k=1,\dots,j-1$. Since $E_{j-1}S_{b_u,j}(A_{b_u..j})f_{\mu,\lambda}=0$, the inductive hypothesis asserting that the current lemma is true for smaller values of $|\bar{A}|$ implies that $(b_u..j)\setminus A_{b_u..j}$ satisfies $\bar{\pi}^{\mu,\lambda}_{b_u,j}(v)$ for some v. As a consequence, there is a weakly increasing injection $e_k:[b_u..c_u]\cup\dots\cup[b_{u+w-1}..c_{u+w-1}]\to[b_u..b_{u+w}-1)$ such that $B^{\mu,\lambda,k}(x,d_k(x))=0$ for any admissible x (here $w=v-1+\delta_{b_u=c_u}$ and $b_{N+1}=j+1$). The inductive hypothesis asserting that property (10) holds for $u+w$ allows us to extend e_k to the required injection d_k. Thus property (10) is proved.

Take any $k=i,\dots,j-1$. Applying property (10) for $u=1$, the fact that x_k follows from x_j, and Proposition 8, we get

$$0=K^{\mu,\lambda}(x_k)=K^{\mu,\lambda,k}_{i,j}(A)=\prod_{t\in[i..j)\setminus\operatorname{Im}d_k}B^{\mu,\lambda,k}(i,t).$$

Therefore, there is $t'\in[i..j)\setminus\operatorname{Im}d_k$ such that $B^{\mu,\lambda,k}(i,t')=0$. Putting $\theta_k(t)=d_k(t)$ for $t\in[b_1..c_1]\cup\dots\cup[b_N..c_N]$ and $\theta_k(i)=t'$, we get a map required in Definition 14. This fact together with Remark 1 shows that $\bar{A}$ satisfies $\bar{\pi}^{\mu,\lambda}_{i,j}(N+1)$, contrary to assumption. $\square$

Following [4], we introduce the following sets:

$$\begin{aligned} \mathfrak{C}^{\mu}(i,j) &:= \{a : i < a < j, a - i + \mu_i - \mu_a \equiv 0 \pmod p\}, \\ \mathfrak{B}^{\mu,\lambda,k}(i,j) &:= \{a : i \leqslant a < j, B^{\mu,\lambda,k}(i,a) = 0\}. \end{aligned}$$

We shall abbreviate $B^{\mu}(i,a) = B^{\mu,\mu}(i,a)$ and $\mathfrak{B}^{\mu}(i,j) = \mathfrak{B}^{\mu,\mu}(i,j)$. It follows from Remark 1 that

$$\mathfrak{B}^{\mu,\lambda,k}(i,j) = \mathfrak{B}^{\mu,\lambda}(i,k) \cup \big(\mathfrak{B}^{\mu}(i,j) \cap [k..j)\big) \tag{11}$$

Theorem 16. *Let $1 \leqslant i < j-1 < n-1$.*

(i) *Let $A \subset (i..j)$. Then $S_{i,j}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector if and only if $j-1 \in A$, for each $k = 1, \ldots, j-1$ there is a weakly increasing injection $\theta_k : [i..j) \setminus A \to [i..j)$ such that $B^{\mu,\lambda,k}(x, \theta_k(x)) = 0$ for any admissible x and there is a weakly increasing injection $d : (i..j) \setminus A \to (i..j)$ such that $B^{\mu,\lambda}(x, d(x)) = 0$ for any admissible x and $B^{\mu,\lambda}(i,t) \neq 0$ for any $t \in [i..j) \setminus \operatorname{Im} d$.*

(ii) *There is some $A \subset (i..j)$ such that $S_{i,j}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector if and only if there are a weakly decreasing injection $\varepsilon : \mathfrak{B}^{\mu,\lambda}(i,j) \to \mathfrak{C}^{\mu}(i,j-1)$ and weakly increasing injections $\theta_k : \{i\} \cup \operatorname{Im}\varepsilon \to \mathfrak{B}^{\mu,\lambda,k}(i,j)$ for any $k = 1, \ldots, j-1$.*

(iii) *There is some $A \subset (i..j)$ such that $S_{i,j}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector if and only if $j-1 \in \mathfrak{B}^{\mu}(i,j)$ (i.e. $B^{\mu}(i,j-1) = 0$), $j-1 \notin \mathfrak{B}^{\mu,\lambda}(i,j)$ (i.e. $B^{\mu,\lambda}(i,j-1) \neq 0$), there are a weakly decreasing and a weakly increasing injections from $\mathfrak{B}^{\mu,\lambda}(i,j-1)$ to $\mathfrak{C}^{\mu}(i,j-1)$ and to $\mathfrak{B}^{\mu}(i,j-1)$ respectively.*

Proof. (i) It is clear from [1, 4.11(ii)], Theorems 12 and 15 and Proposition 8 that $S_{i,j}(A)f_{\mu,\lambda}$ is a non-zero $U(n-1)$-high weight vector for such A.

Conversely, let $S_{i,j}(A)f_{\mu,\lambda}$ be a non-zero $U(n-1)$-high weight vector. Suppose that $j-1 \notin A$. Since $\big((i,j,j,A)\big) \xrightarrow{j-1} \big((i,j-1,j,A)\big)$, we have by Theorem 6 that $K_{i,j}^{\mu,\lambda,j}(A) \neq 0$ and $K_{i,j}^{\mu,\lambda,j-1}(A) = 0$. However, it is impossible since by Lemma 9(i) and Remark 1, we have $K_{i,j}^{\mu,\lambda,j}(A) = K_{i,j-1}^{\mu,\lambda}(A) = K_{i,j}^{\mu,\lambda,j-1}(A)$. Thus we have proved that $j-1 \in A$.

Arguing as in the "only if part" of Theorem 12, we get that for each $k = 1, \ldots, j$ there is a weakly increasing injection $d_k : (i..j) \setminus A \to (i..j)$ such that $B^{\mu,\lambda,k}(x, d_k(x)) = 0$ for any admissible x. By Theorem 6, we have $K_{i,j}^{\mu,\lambda}(A) \neq 0$. Hence by Proposition 8, we have $B^{\mu,\lambda}(i,t) \neq 0$ for any $t \in [i..j) \setminus \operatorname{Im} d_k$. Since each sequence $\big((i,k,j,A)\big)$, where $k = i, \ldots, j-1$, follows from $\big((i,j,j)\big)$ we have $K_{i,j}^{\mu,\lambda,k}(A) = 0$ for each $k = 1, \ldots, j-1$. Applying Proposition 8, we get the required maps $\theta_1, \ldots, \theta_{j-1}$.

(ii) If ε and $\theta_1, \ldots, \theta_{j-1}$ are such injections, then it suffices to put $A = (i..j) \setminus \operatorname{Im}\varepsilon$, take for d the inverse map of ε and apply part (i).

Conversely, let $S_{i,j}(A)f_{\mu,\lambda}$ be a non-zero $U(n-1)$-high weight vector for some $A \subset (i..j)$ and let d and $\theta_1, \ldots, \theta_{j-1}$ be injections, whose existence is claimed by part (i). Note that the following two facts: $\mathfrak{B}^{\mu,\lambda}(i,j) \subset \operatorname{Im} d$; $d(x) \in \mathfrak{B}^{\mu,\lambda}(i,j)$ implies $x \in \mathfrak{C}^{\mu}(i,j)$. Now we define $\varepsilon(d(x)) := x$ for

$x \in d^{-1}(\mathfrak{B}^{\mu,\lambda}(i,j))$. Observing that $\operatorname{Im}\varepsilon = d^{-1}(\mathfrak{B}^{\mu,\lambda}(i,j)) \subset (\mathfrak{C}^{\mu}(i,j-1)) \cap ((i..j) \setminus A)$ completes the proof.

(iii) Let $j-1 \in \mathfrak{B}^{\mu}(i,j)$, $j-1 \notin \mathfrak{B}^{\mu,\lambda}(i,j)$ and $\varepsilon : \mathfrak{B}^{\mu,\lambda}(i,j-1) \to \mathfrak{C}^{\mu}(i,j-1)$ and $\tau : \mathfrak{B}^{\mu,\lambda}(i,j-1) \to \mathfrak{B}^{\mu}(i,j-1)$ be a weakly decreasing and a weakly increasing injections respectively. We have $\mathfrak{B}^{\mu,\lambda}(i,j) = \mathfrak{B}^{\mu,\lambda}(i,j-1)$. Thus it remains to define injections $\theta_1,\dots,\theta_{j-1}$. For $x \in \{i\} \cup \operatorname{Im}\varepsilon$ and $k = 1,\dots,j-1$, we put

$$\theta_k(x) = \begin{cases} j-1 & \text{if } x = i; \\ \varepsilon^{-1}(x) & \text{if } i < x \text{ and } \varepsilon^{-1}(x) < k; \\ \tau(\varepsilon^{-1}(x)) & \text{if } i < x \text{ and } \varepsilon^{-1}(x) \geqslant k; \end{cases}$$

One can easily verify with the help of (11) that $\varepsilon, \theta_1, \dots, \theta_{j-1}$ thus defined satisfy the conditions from part (ii).

Conversely, let $\varepsilon, \theta_1, \dots, \theta_{j-1}$ be as in part (ii). For $k = 1,\dots,j-1$, we have $|\mathfrak{B}^{\mu,\lambda,k}(i,j)| \geqslant |\operatorname{Im}\theta_k| = |\{i\} \cup \operatorname{Im}\varepsilon| = 1 + |\mathfrak{B}^{\mu,\lambda}(i,j)|$. Taking $k = j-1$ and applying (11), we get

$$\begin{aligned} &|\mathfrak{B}^{\mu,\lambda}(i,j-1)| + |\mathfrak{B}^{\mu}(i,j) \cap \{j-1\}| = |\mathfrak{B}^{\mu,\lambda,j-1}(i,j)| \\ &\geqslant 1 + |\mathfrak{B}^{\mu,\lambda}(i,j)| = 1 + |\mathfrak{B}^{\mu,\lambda}(i,j-1)| + |\mathfrak{B}^{\mu,\lambda}(i,j) \cap \{j-1\}|. \end{aligned}$$

Hence $|\mathfrak{B}^{\mu}(i,j) \cap \{j-1\}| = 1 + |\mathfrak{B}^{\mu,\lambda}(i,j) \cap \{j-1\}|$, whence $j-1 \in \mathfrak{B}^{\mu}(i,j)$ and $j-1 \notin \mathfrak{B}^{\mu,\lambda}(i,j)$. Next for any $k = 1,\dots,j-1$, we have

$$\begin{aligned} &1 + |\mathfrak{B}^{\mu,\lambda}(i,k)| + |\mathfrak{B}^{\mu,\lambda}(i,j-1) \cap [k..j-1)| = 1 + |\mathfrak{B}^{\mu,\lambda}(i,j)| \leqslant |\mathfrak{B}^{\mu,\lambda,k}(i,j)| \\ &= |\mathfrak{B}^{\mu,\lambda}(i,k)| + |\mathfrak{B}^{\mu}(i,j-1) \cap [k..j-1)| + 1. \end{aligned}$$

Hence $|\mathfrak{B}^{\mu,\lambda}(i,j-1) \cap [k..j-1)| \leqslant |\mathfrak{B}^{\mu}(i,j-1) \cap [k..j-1)|$ for any $k = 1,\dots,j-1$ and by [1, 2.2] there is a weakly increasing injection $\tau : \mathfrak{B}^{\mu,\lambda}(i,j-1) \to \mathfrak{B}^{\mu}(i,j-1)$. □

Theorem 17. *Part* (iii) *of Theorem 16 remains true for* $1 < j = i+1 < n$.

Proof. Indeed, $S_{i,i+1}(\varnothing) = F_{i,i+1}$ is a non-zero $U(n-1)$-high weight vector if and only if $\mu_i - \lambda_{i+1} \not\equiv 0 \pmod p$ and $\mu_i - \mu_{i+1} \equiv 0 \pmod p$. Taking into account $\mathfrak{B}^{\mu,\lambda}(i,j-1) = \varnothing$, $B^{\mu,\lambda}(i,j-1) = \mu_i - \lambda_{i+1} + p\mathbb{Z}$ and $B^{\mu}(i,j-1) = \mu_i - \mu_{i+1} + p\mathbb{Z}$, we obtain the required result. □.

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